

# A characterization of hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem

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## Abstract

For  $k \geq 2$ , let  $H$  be a  $k$ -uniform hypergraph on  $n$  vertices and  $m$  edges. The transversal number  $\tau(H)$  of  $H$  is the minimum number of vertices that intersect every edge. Chvátal and McDiarmid [Combinatorica 12 (1992), 19–26] proved that  $\tau(H) \leq (n + \lfloor \frac{k}{2} \rfloor m) / (\lfloor \frac{3k}{2} \rfloor)$ . When  $k = 3$ , the connected hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem were characterized by Henning and Yeo [J. Graph Theory 59 (2008), 326–348]. In this paper, we characterize the connected hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem for  $k = 2$  and for all  $k \geq 4$ .

**Keywords:** Transversal; hypergraph; edge coloring; matchings; multigraph.

**AMS subject classification:** 05C65

## 1 Introduction

In this paper we continue the study of transversals in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph*  $H = (V, E)$  is a finite set  $V = V(H)$  of elements, called *vertices*, together with a finite multiset  $E = E(H)$  of subsets of  $V$ , called *hyperedges* or simply *edges*.

A  $k$ -*edge* in  $H$  is an edge of size  $k$ . The hypergraph  $H$  is said to be  $k$ -*uniform* if every edge of  $H$  is a  $k$ -edge. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. The *degree* of a vertex  $v$  in  $H$ , denoted by  $d_H(v)$  or simply by  $d(v)$  if  $H$  is clear from the context, is the number of edges of  $H$  which contain  $v$ . The minimum and maximum degrees among the vertices of  $H$  is denoted by  $\delta(H)$  and  $\Delta(H)$ , respectively.

Two vertices  $x$  and  $y$  of  $H$  are *adjacent* if there is an edge  $e$  of  $H$  such that  $\{x, y\} \subseteq e$ . The *neighborhood* of a vertex  $v$  in  $H$ , denoted  $N_H(v)$  or simply  $N(v)$  if  $H$  is clear from the context, is the set of all vertices different from  $v$  that are adjacent to  $v$ . Two vertices  $x$  and  $y$  of  $H$  are *connected* if there is a sequence  $x = v_0, v_1, v_2, \dots, v_k = y$  of vertices of  $H$  in

which  $v_{i-1}$  is adjacent to  $v_i$  for  $i = 1, 2, \dots, k$ . A *connected hypergraph* is a hypergraph in which every pair of vertices are connected. A maximal connected subhypergraph of  $H$  is a *component* of  $H$ . Thus, no edge in  $H$  contains vertices from different components.

If  $H$  denotes a hypergraph and  $X$  denotes a subset of vertices in  $H$ , then  $H - X$  will denote that hypergraph obtained from  $H$  by removing the vertices  $X$  from  $H$ , removing all hyperedges that intersect  $X$  and removing all resulting isolated vertices, if any. If  $X = \{x\}$ , we simply denote  $H - X$  by  $H - x$ . We remark that in the literature this is sometimes denoted by *strongly deleting* the vertices in  $X$ .

A subset  $T$  of vertices in a hypergraph  $H$  is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if  $T$  has a nonempty intersection with every edge of  $H$ . The *transversal number*  $\tau(H)$  of  $H$  is the minimum size of a transversal in  $H$ . A transversal of size  $\tau(H)$  is called a  $\tau(H)$ -set. Transversals in hypergraphs are well studied in the literature (see, for example, [1, 2, 3, 4, 5, 8, 10]). Chvátal and McDiarmid [1] established the following upper bound on the transversal number of a uniform hypergraphs in terms of its order and size.

**Chvátal-McDiarmid Theorem.** *For  $k \geq 2$ , if  $H$  is a  $k$ -uniform hypergraph on  $n$  vertices with  $m$  edges, then*

$$\tau(H) \leq \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor}.$$

As a special case of the Chvátal-McDiarmid Theorem when  $k = 3$ , we have that if  $H$  is a 3-uniform hypergraph on  $n$  vertices with  $m$  edges, then  $\tau(H) \leq (n + m)/4$ . This bound was independently established by Tuza [11] and a short proof of this result was also given by Thomassé and Yeo [10]. The extremal connected hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem when  $k = 3$  were characterized by Henning and Yeo [3]. Their characterization showed that there are three infinite families of extremal connected hypergraphs, as well as two special hypergraphs, one of order 7 and the other of order 8.

Our aim in this paper is to characterize the connected hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem for  $k = 2$  and for all  $k \geq 4$ . For this purpose we define two special families of hypergraphs.

## 1.1 Special Families of Hypergraphs

For  $k \geq 2$ , let  $E_k$  denote the  $k$ -uniform hypergraph on  $k$  vertices with exactly one edge. The hypergraph  $E_4$  is illustrated in Figure 1.

For  $k \geq 2$ , a *generalized triangle*  $T_k$  is defined as follows. Let  $A$ ,  $B$ ,  $C$  and  $D$  be vertex-disjoint sets of vertices with  $|A| = \lceil k/2 \rceil$ ,  $|B| = |C| = \lfloor k/2 \rfloor$  and  $|D| = \lceil k/2 \rceil - \lfloor k/2 \rfloor$ . In particular, if  $k$  is even, the set  $D = \emptyset$ , while if  $k$  is odd, the set  $D$  consist of a singleton vertex. Let  $T_k$  denote the  $k$ -uniform hypergraph with  $V(T_k) = A \cup B \cup C \cup D$  and with  $E(T_k) = \{e_1, e_2, e_3\}$ , where  $V(e_1) = A \cup B$ ,  $V(e_2) = A \cup C$ , and  $V(e_3) = B \cup C \cup D$ . The hypergraphs  $T_4$  and  $T_5$  are illustrated in Figure 1.

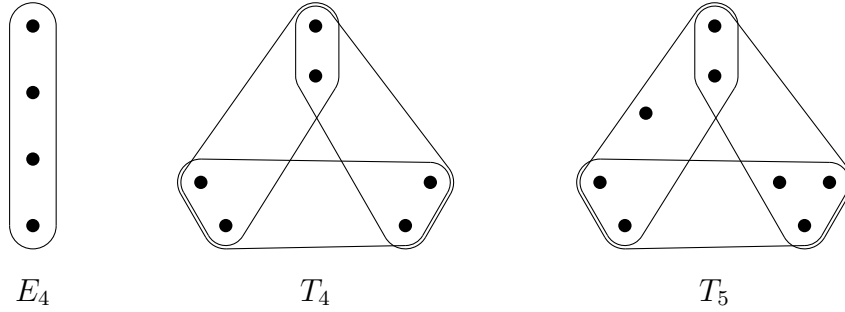


Figure 1: The hypergraphs  $E_4$ ,  $T_4$ , and  $T_5$

## 2 Main Result

We shall prove:

**Theorem 1** *For  $k = 2$  or  $k \geq 4$ , let  $H$  be a connected  $k$ -uniform hypergraph on  $n$  vertices and  $m$  edges. Then,*

$$\tau(H) \leq \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor}$$

*with equality if and only if  $H = E_k$  or  $H = T_k$ .*

We proceed as follows. We first recall some important results on edge colorings of multigraphs in Section 3. Thereafter we establish a key theorem about matchings in multigraphs in Section 4. Finally in Section 5 we present a proof of Theorem 1 using an interplay between transversals in hypergraphs and matchings in multigraphs.

## 3 Edge Colorings of Multigraphs

Let  $G$  be a multigraph. An *edge coloring* of  $G$  is an assignment of colors to the edges of  $G$  such that adjacent edges receive different colors. The minimum number of colors needed for an edge coloring is called the chromatic index of the multigraph, denoted  $\chi'(G)$ . The edge-multiplicity of an edge  $e = uv$ , written  $\mu(uv)$ , is the number of edges joining  $u$  and  $v$ . In his study of electrical networks, Shannon [9] established the following upper bound on the chromatic index of a multigraph.

**Shannon's Theorem.** *If  $G$  is a multigraph, then  $\chi'(G) \leq \lfloor 3\Delta(G)/2 \rfloor$ .*

For  $d \geq 2$ , a *Shannon multigraph of degree  $d$*  is a multigraph on three vertices, with one pair of vertices joined by  $\lceil d/2 \rceil$  edges and the other two pairs joined by  $\lfloor d/2 \rfloor$  edges. Thus for fixed  $d$ , all Shannon multigraphs of degree  $d$  are isomorphic to the multigraph  $G$  with

vertex set  $V(G) = \{x, y, z\}$  and with  $\mu(xy) = \lfloor d/2 \rfloor$ ,  $\mu(xz) = \lfloor d/2 \rfloor$  and  $\mu(yz) = \lceil d/2 \rceil$ . A characterization of multigraphs achieving the upper bound in Shannon's Theorem when the maximum degree is at least 4 was given by Vizing [12].

**Vizing's Theorem.** *If  $G$  is a connected multigraph with  $\Delta(G) \geq 4$  and  $\chi'(G) = \lfloor 3\Delta(G)/2 \rfloor$ , then  $G$  contains a Shannon multigraph of degree  $\Delta(G)$  as a submultigraph.*

We remark that the maximum degree condition in Vizing's Theorem is essential. For example, if  $G$  is a connected multigraph with  $\Delta(G) = 2$  and  $\chi'(G) = \lfloor 3\Delta(G)/2 \rfloor = 3$ , then  $G$  need not contain a Shannon multigraph of degree  $\Delta(G)$  as a subgraph as may be seen by simply taking  $G$  to be an odd cycle of length at least 5.

## 4 Matchings in Multigraphs

Let  $G$  be a multigraph. Two edges in  $G$  are *independent* if they are not adjacent in  $G$ . A set of pairwise independent edges of  $G$  is called a *matching* in  $G$ , while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum matching of  $G$  is called the *matching number* of  $G$  which we denote by  $\alpha'(G)$ . Our key matching theorem characterizes connected multigraphs with small matching number determined by Shannon's Theorem.

**Theorem 2** *For  $d \geq 4$ , let  $G$  be a connected multigraph of size  $m$  with  $\Delta(G) \leq d$ . Then,  $\alpha'(G) \geq m / \lfloor 3d/2 \rfloor$ , with equality if and only if either  $m = 0$  or  $G$  is a Shannon multigraph of degree  $d$ .*

**Proof.** Let  $\mathcal{C}$  be an arbitrary edge coloring of the edges of  $G$  using  $\chi'(G)$  colors. The matching number of  $G$  is at least the cardinality of a maximum edge color class in  $\mathcal{C}$ , and so, by Shannon's Theorem,

$$\alpha'(G) \geq \frac{m}{\chi'(G)} \geq \frac{m}{\left\lfloor \frac{3\Delta(G)}{2} \right\rfloor} \geq \frac{m}{\left\lfloor \frac{3d}{2} \right\rfloor},$$

which establishes the desired lower bound. Suppose that  $\alpha'(G) = m / \lfloor 3d/2 \rfloor$  and  $m \geq 1$ . Then we must have equality throughout the above inequality chain. Thus,  $\Delta(G) = d$ ,  $\chi'(G) = \lfloor 3d/2 \rfloor$  and  $\alpha'(G) = m / \chi'(G)$ . In particular, since  $\mathcal{C}$  is an arbitrary  $\chi'(G)$ -edge coloring, the edge color classes in every  $\chi'(G)$ -edge coloring have the same cardinality. Equivalently, the edge color classes in  $\mathcal{C}$  are balanced. Since  $\chi'(G) = \lfloor 3d/2 \rfloor$ , Vizing's Theorem implies that  $G$  contains a Shannon multigraph,  $M$  say, of degree  $d$  as a submultigraph.

If  $d$  is even, then every vertex of  $M$  has degree  $d$  in  $M$ . Since  $\Delta(G) = d$ , the Shannon multigraph  $M$  cannot be a proper submultigraph of the connected multigraph  $G$ , implying that  $G = M$ . Hence if  $d$  is even, then  $G$  is a Shannon multigraph of degree  $d$ . Therefore we may assume that  $d$  is odd, for otherwise the desired result holds.

Since  $d$  is odd,  $d \geq 5$  and one pair of vertices in  $M$  is joined by  $(d+1)/2$  edges and the other two pairs are joined by  $(d-1)/2$  edges. Thus two vertices in  $M$  have degree  $d$  in  $M$  and one vertex,  $x$  say, of  $M$  has degree  $d-1$  in  $M$ . Assume that  $M$  is a proper submultigraph of  $G$ . Since  $\Delta(G) = d$ , the vertex  $x$  is adjacent in  $G$  to exactly one vertex  $v \notin V(M)$ . Since  $xv$  is a bridge in  $G$ , the edge  $xv$  cannot belong to a submultigraph of  $G$  that is isomorphic to a Shannon multigraph of degree  $d$ . Thus all submultigraphs of  $G$  that are isomorphic to a Shannon multigraph of degree  $d$  are vertex-disjoint.

Let  $G'$  be the multigraph that arises from  $G$  by deleting every edge from  $G$  that belongs to a submultigraph of  $G$  that is isomorphic to a Shannon multigraph of degree  $d$ . Then,  $\Delta(G') \leq \Delta(G) = d$  and, by construction,  $G'$  does not contain a submultigraph of  $G$  that is isomorphic to a Shannon multigraph of degree  $d$ . Since  $xv \in E(G')$ , the multigraph  $G'$  has at least one edge. By Shannon's Theorem and Vizing's Theorem, we deduce that  $\chi'(G) < \lfloor 3d/2 \rfloor$ .

Let  $\mathcal{C}'$  be a  $\chi'(G')$ -edge coloring of the edges of  $G'$ . By construction, every submultigraph of the connected multigraph  $G$  that is isomorphic to a Shannon multigraph of degree  $d$  contains exactly one vertex that is incident with an edge of  $G'$ . Since  $\chi'(G) < \lfloor 3d/2 \rfloor$ , the coloring  $\mathcal{C}'$  can therefore be extended to a  $\lfloor 3d/2 \rfloor$ -edge coloring  $\mathcal{C}^*$  of  $G$ . Since  $\mathcal{C}'$  colors the edges of  $G'$  with fewer than  $\lfloor 3d/2 \rfloor$  colors,  $\mathcal{C}^*$  is a  $\chi'(G)$ -edge coloring of the edges of  $G$  with at least two edge color classes having different cardinality. This contradicts our earlier observation that the edge color classes in every  $\chi'(G)$ -edge coloring have the same cardinality. Therefore,  $M$  is not a proper submultigraph of the connected multigraph  $G$ , implying that  $G = M$ . Hence if  $d$  is odd, then  $G$  is a Shannon multigraph of degree  $d$ .

Conversely, if  $G$  is a Shannon multigraph of degree  $d$ , then  $m = \lfloor 3d/2 \rfloor$  and  $\alpha'(G) = 1$ , implying that  $\alpha'(G) = m / \lfloor 3d/2 \rfloor$ .  $\square$

We close this section by recalling Hall's Matching Theorem due to König [7] and Hall [6].

**Hall's Matching Theorem.** *Let  $G$  be a bipartite graph with partite sets  $X$  and  $Y$ . Then  $X$  can be matched to a subset of  $Y$  if and only if  $|N(S)| \geq |S|$  for every nonempty subset  $S$  of  $X$ .*

## 5 Proof of Main Result

We shall need the following properties of special hypergraphs defined in Section 1.1.

**Observation 3** *Let  $k \geq 2$  and let  $H = E_k$  or  $H = T_k$  and let  $H$  have  $n$  vertices and  $m$  edges. Then the following holds.*

- (a) *If  $H = E_k$ , then  $\tau(H) = 1$ .*
- (b) *If  $H = T_k$ , then  $\tau(H) = 2$ .*
- (c)  *$\tau(H) = (n + \lfloor \frac{k}{2} \rfloor m) / \lfloor \frac{3k}{2} \rfloor$ .*
- (d) *Every vertex in  $H$  belongs to some  $\tau(H)$ -set.*

We are now in a position to prove our main result. Recall the statement of Theorem 1.

**Theorem 1.** *For  $k = 2$  or  $k \geq 4$ , let  $H$  be a connected  $k$ -uniform hypergraph on  $n$  vertices and  $m$  edges. Then,*

$$\tau(H) \leq \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor}$$

*with equality if and only if  $H = E_k$  or  $H = T_k$ .*

**Proof.** The upper bound on  $\tau(H)$  is a restatement of the Chvátal-McDiarmid Theorem. We only need prove that  $\tau(H) = (n + \lfloor \frac{k}{2} \rfloor m) / \lfloor \frac{3k}{2} \rfloor$  if and only if  $H = E_k$  or  $H = T_k$ . If  $H = E_k$  or  $H = T_k$ , then by Observation 3,  $\tau(H) = (n + \lfloor \frac{k}{2} \rfloor m) / \lfloor \frac{3k}{2} \rfloor$ , as desired.

To prove the converse, suppose that  $\tau(H) = (n + \lfloor \frac{k}{2} \rfloor m) / \lfloor \frac{3k}{2} \rfloor$ , where  $k = 2$  or  $k \geq 4$ . We proceed by induction on the order  $n$  to show that  $H = E_k$  or  $H = T_k$ . If  $m = 0$ , then  $\tau(H) = 0 < n / \lfloor \frac{3k}{2} \rfloor$ , a contradiction. Hence  $m \geq 1$ , and so  $n \geq k$ . If  $n = k$ , then  $H = E_k$ , and we are done. This establishes the base case. Let  $n \geq k + 1$  and let  $H$  be a connected  $k$ -uniform hypergraph on  $n$  vertices and  $m$  edges, and assume that the desired result holds for all connected  $k$ -uniform hypergraph on fewer than  $n$  vertices.

**Claim A**  $\delta(H) \geq 1$ .

**Proof.** Suppose that  $\delta(H) = 0$ . Let  $F$  be obtained from  $H$  by deleting all isolated vertices. Let  $F$  have  $n_F$  vertices and  $m_F$  edges. Then,  $n_F \leq n - 1$  and  $m_F = m$ . Every transversal in  $H'$  is a transversal in  $H$ , and so  $\tau(H) \leq \tau(H')$ . By the Chvátal-McDiarmid Theorem, we have that

$$\tau(H) \leq \tau(H') \leq \frac{n_F + \lfloor \frac{k}{2} \rfloor m_F}{\lfloor \frac{3k}{2} \rfloor} < \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor},$$

a contradiction. Hence,  $\delta(H) \geq 1$ . ( $\square$ )

Let  $v$  be a vertex of maximum degree  $\Delta(H)$  in  $H$  and let  $H' = H - v$  have  $n'$  vertices and  $m'$  edges. Then,  $H'$  is a  $k$ -uniform hypergraph. Every transversal in  $H'$  can be extended to a transversal in  $H$  by adding to it the vertex  $v$ , and so  $\tau(H) \leq \tau(H') + 1$ . Recall that  $k = 2$  or  $k \geq 4$ .

**Claim B** *If  $k$  is even, then  $\Delta(H) \leq 2$ , while if  $k$  is odd, then  $\Delta(H) \leq 3$ .*

**Proof.** Suppose first that  $k$  is even and  $\Delta(H) \geq 3$ . Then,  $n' \leq n - 1$  and  $m' \leq m - 3$ . Since  $k$  is even, we have by the Chvátal-McDiarmid Theorem that

$$\tau(H) \leq \tau(H') + 1 \leq \frac{n' + \lfloor \frac{k}{2} \rfloor m'}{\lfloor \frac{3k}{2} \rfloor} + 1 \leq \frac{n + \lfloor \frac{k}{2} \rfloor m - 1}{\lfloor \frac{3k}{2} \rfloor} < \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor},$$

a contradiction. Hence if  $k$  is even, then  $\Delta(H) \leq 2$ . Suppose next that  $k$  is odd and  $\Delta(H) \geq 4$ . Then,  $n' \leq n - 1$  and  $m' \leq m - 4$ . Since  $k \geq 5$  is odd, we have by the

Chvátal-McDiarmid Theorem that

$$\tau(H) \leq \tau(H') + 1 \leq \frac{n' + \lfloor \frac{k}{2} \rfloor m'}{\lfloor \frac{3k}{2} \rfloor} + 1 \leq \frac{(n-1) + \lfloor \frac{k}{2} \rfloor (m-4)}{\lfloor \frac{3k}{2} \rfloor} + 1 < \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor},$$

a contradiction. Hence if  $k$  is odd, then  $\Delta(H) \leq 3$ .  $\square$

**Claim C** *If  $k = 2$ , then  $H$  is a generalized triangle  $T_2$ .*

**Proof.** Suppose that  $k = 2$ , and so  $H$  is a graph and  $\tau(H) = (n+m)/3$ . By Claim A and Claim B, we have that  $\delta(H) \geq 1$  and  $\Delta(H) \leq 2$ . Thus,  $H$  is a path or a cycle. If  $H$  is a path on  $n \geq 2$  vertices, then  $(2n-1)/3 = (n+m)/3 = \tau(H) = \lfloor n/2 \rfloor$ , implying that  $n = 2$  and  $H = E_2$ . However this contradicts the fact that  $n \geq k+1$ . Hence,  $H$  is a cycle on  $n \geq 3$  vertices. Thus,  $2n/3 = (n+m)/3 = \tau(H) = \lceil n/2 \rceil$ , implying that  $n = 3$  and  $H$  is a generalized triangle  $T_2$ .  $\square$

In what follows we may assume that  $k \geq 4$ , for otherwise the desired result follows by Claim C.

**Claim D** *If  $\Delta(H) \leq 2$ , then  $H = T_k$ .*

**Proof.** Suppose that  $\Delta(H) \leq 2$ . For  $i = 1, 2$ , let  $n_i$  be the number of vertices of degree  $i$  in  $H$ . By Claim A,  $\delta(H) \geq 1$  and so  $n_1 + n_2 = n$ . By the  $k$ -uniformity of  $H$  we have that  $n_1 + 2n_2 = km$ , or, equivalently,  $n_2 = km - n$ . We now consider the multigraph  $G$  whose vertices are the edges of  $H$  and whose edges correspond to the  $n_2$  vertices of degree 2 in  $H$ : if a vertex of  $H$  is contained in the edges  $e$  and  $f$  of  $H$ , then the corresponding edge of  $G$  joins vertices  $e$  and  $f$  of  $G$ . Since  $H$  is  $k$ -uniform and  $\Delta(H) \leq 2$ , the maximum degree in  $G$  is at most  $k$ . Further since  $H$  is connected, so too is  $G$ .

Let  $M$  be a maximum matching in  $G$ , and so by Theorem 2,  $|M| = \alpha'(G) \geq n_2 / \lfloor 3k/2 \rfloor$ . Let  $S$  be the set of vertices of  $H$  that correspond to the set of edges  $M$  in  $G$ . Then,  $S$  is an independent set in  $H$  and every vertex in  $S$  has degree 2 in  $H$ . By the maximality of  $M$ , we note that the set of edges in  $H$  that do not intersect  $S$  are vertex-disjoint. Let  $S'$  be a set of vertices in  $H$  that consists of exactly one vertex from every edge of  $H$  that does not intersect  $S$ . Then,  $|S'| = m - 2|S|$  and the set  $S \cup S'$  is a transversal in  $H$ . Thus,  $\tau(H) \leq |S| + |S'| = m - |S| = m - |M|$ . Hence,

$$\frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor} = \tau(H) \leq m - |M| \leq m - \frac{n_2}{\lfloor \frac{3k}{2} \rfloor} = m - \frac{km - n}{\lfloor \frac{3k}{2} \rfloor} = \frac{\lfloor \frac{k}{2} \rfloor m + n}{\lfloor \frac{3k}{2} \rfloor}.$$

Consequently, we must have equality throughout the above inequality chain. In particular,  $\alpha'(G) = |M| = n_2 / \lfloor 3k/2 \rfloor$ . Thus by Theorem 2, either  $n_2 = 0$  or  $G$  is a Shannon multigraph of degree  $k$ . If  $n_2 = 0$ , then  $n = n_1$ , implying by the connectivity of  $H$  that  $H = E_k$ .

However this contradicts the fact that  $n \geq k + 1$ . Hence,  $G$  is a Shannon multigraph of degree  $k$ , implying that  $H$  is a generalized triangle  $T_k$ .  $\square$

By Claim D, if  $\Delta(H) \leq 2$ , then  $H$  is a generalized triangle  $T_k$ , and we are done. Hence we may assume in what follows that  $\Delta(H) \geq 3$ . By Claim B,  $k \geq 5$  is odd and  $\Delta(H) = 3$ . We now prove a series of claims that culminate in a contradiction.<sup>1</sup>

**Claim E** *The following hold in the hypergraph  $H$ .*

- (a)  $\tau(H) = \tau(H') + 1$ .
- (b)  $n' = n - 1$ .
- (c) *Every component of  $H'$  is either  $E_k$  or  $T_k$ .*

**Proof.** Since  $\Delta(H) = 3$ , we note that  $n' \leq n - 1$  and  $m' = m - 3$ . Since  $k$  is odd, we have by the Chvátal-McDiarmid Theorem that

$$\tau(H) \leq \tau(H') + 1 \leq \frac{n' + \lfloor \frac{k}{2} \rfloor m'}{\lfloor \frac{3k}{2} \rfloor} + 1 \leq \frac{(n - 1) + \lfloor \frac{k}{2} \rfloor (m - 3)}{\lfloor \frac{3k}{2} \rfloor} + 1 = \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor},$$

Since  $\tau(H) = (n + \lfloor \frac{k}{2} \rfloor m) / \lfloor \frac{3k}{2} \rfloor$ , we must have equality throughout the above inequality chain, implying that  $\tau(H) = \tau(H') + 1$ ,  $\tau(H') = (n' + \lfloor \frac{k}{2} \rfloor m') / \lfloor \frac{3k}{2} \rfloor$  and  $n' = n - 1$ . Applying the inductive hypothesis to every component of  $H'$ , we have that every component of  $H'$  is either  $E_k$  or  $T_k$ .  $\square$

By Claim E(c) every component of  $H'$  is either  $E_k$  or  $T_k$ . By Observation 3, every component of  $H'$  that is  $E_k$  or  $T_k$  contributes 1 or 2, respectively, to  $\tau(H')$ .

Let  $e_1, e_2, e_3$  be the three edges that contain the vertex  $v$  in  $H$  and let  $E_v = \{e_1, e_2, e_3\}$ . By Claim E(b),  $n' = n - 1$ , which implies that  $|V(e) \cap V(H')| = k - 1$  for each edge  $e \in E_v$ .

**Claim F** *Let  $C$  be a component of  $H'$  that is a generalized triangle  $T_k$ . If  $|V(C) \cap V(e_1)| \geq 2$ ,  $|V(C) \cap V(e_2)| \geq 2$  and  $|V(C) \cap V(e_3)| \leq k - 2$ , then  $|V(C) \cap V(e_1)| + |V(C) \cap V(e_2)| \leq (k + 1)/2$ .*

**Proof.** Assume, to the contrary, that  $|V(C) \cap V(e_1)| + |V(C) \cap V(e_2)| > (k + 1)/2$ . Since  $|V(C) \cap V(e_3)| \leq k - 2$ , there is a vertex  $u_3 \in V(e_3) \setminus (V(C) \cup \{v\})$ . If there is a vertex  $u_1 \in V(C) \cap V(e_1) \cap V(e_2)$ , then by Observation 3(d) there is a  $\tau(H')$ -set  $T$  that contains both  $u_1$  and  $u_3$ . Since  $\{u_1, u_3\}$  intersects all three edges that contain  $v$  in  $H$ , the set  $T$  is a transversal of  $H$ , and so  $\tau(H) \leq |T| = \tau(H')$ , contradicting Claim E(a). Hence,  $V(C) \cap V(e_1) \cap V(e_2) = \emptyset$ .

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<sup>1</sup>We remark that if we allow  $k = 3$ , then it is indeed possible that  $\Delta(H) = 3$ . The current proof technique therefore fails in this special case when  $k = 3$  since we are then unable to associate a multigraph with the hypergraph  $H$  as is done in the proof of Claim D. However as remarked earlier, the special case when  $k = 3$  has fortunately been handled in [3].



Since  $|V(C) \cap V(e_1)| \geq 2$  and since there is a unique vertex of  $C$  of degree 1 in  $H'$ , there is a vertex  $u_1 \in V(C) \cap V(e_1)$  of degree 2 in  $H'$ . Let  $f$  be the unique edge of  $C$  that does not contain  $u_1$ . If there is a vertex  $u_2 \in V(f) \cap V(e_2)$ , then  $\{u_1, u_2\}$  is a  $\tau(C)$ -set and, by Observation 3(d), there is a  $\tau(H')$ -set  $T$  that contains the set  $\{u_1, u_2, u_3\}$ . Since  $\{u_1, u_2, u_3\}$  intersects all three edges that contain  $v$  in  $H$ , the set  $T$  is a transversal of  $H$ , and so  $\tau(H) \leq |T| = \tau(H')$ , contradicting Claim E(a). Hence,  $V(f) \cap V(e_2) = \emptyset$ .

Since  $|V(C) \cap V(e_2)| \geq 2$  and since there is a unique vertex of  $C$  of degree 1 in  $H'$ , there is a vertex  $x_2 \in V(C) \cap V(e_2)$  of degree 2 in  $H'$ . By assumption,  $|V(C) \cap V(e_1)| + |V(C) \cap V(e_2)| > (k+1)/2$ . As observed earlier, the edges  $e_1$  and  $e_2$  do not intersect in  $C$  and  $V(f) \cap V(e_2) = \emptyset$ . Since  $|V(C) \setminus V(f)| = (k+1)/2$ , there is therefore a vertex  $x_1 \in V(f) \cap V(e_1)$ . Thus the set  $\{x_1, x_2\}$  is a  $\tau(C)$ -set and, by Observation 3(d), there is a  $\tau(H')$ -set  $T$  that contains the set  $\{x_1, x_2, u_3\}$ . Since  $\{x_1, x_2, u_3\}$  intersects all three edges that contain  $v$  in  $H$ , the set  $T$  is a transversal of  $H$ , and so  $\tau(H) \leq |T| = \tau(H')$ , once again contradicting Claim E(a). Therefore,  $|V(C) \cap V(e_1)| + |V(C) \cap V(e_2)| \leq (k+1)/2$ .  $\square$

**Claim G**  $H'$  is disconnected.

**Proof.** Assume, to the contrary, that  $H'$  is connected. Since  $\Delta(H) = 3$ , the  $k$ -uniformity of  $H$  implies that  $km = \sum_{v \in V(H)} d(v) \leq 3n$ . By Claim E(c),  $H$  is either  $E_k$  or  $T_k$ . Suppose first that  $H = E_k$ . Then,  $n = k+1$  and  $m = 4$ . However  $k \geq 5$ , and so  $km = 4k \geq 3k+5 > 3k+3 = 3n$ , a contradiction. Hence,  $H = T_k$ . Thus,  $n = 3(k+1)/2$  and  $m = 6$ . However  $k \geq 5$ , and so  $km = 6k = 9k/2 + 3k/2 \geq 9k/2 + 15/2 > 9k/2 + 9/2 = 3n$ , once again producing a contradiction. Therefore,  $H'$  is disconnected.  $\square$

**Claim H**  $H'$  has at least three components.

**Proof.** Assume, to the contrary, that  $H'$  has at most two components. Then by Claim G, the hypergraph  $H'$  has exactly two components which we call  $C_1$  and  $C_2$ . As observed earlier,  $|V(e) \cap V(H')| = k-1$  for each edge  $e \in E_v$ . Renaming the components  $C_1$  and  $C_2$  if necessary, we may assume that

$$\sum_{e \in E_v} |V(C_1) \cap V(e)| \geq \frac{3}{2}(k-1) \geq \sum_{e \in E_v} |V(C_2) \cap V(e)| \quad (1)$$

and that if we have equality throughout the Inequality Chain (1), then  $V(C_2)$  intersects at least as many edges of  $E_v$  as  $V(C_1)$  does. Since  $H$  is connected, the vertex  $v$  is adjacent in  $H$  to a vertex from  $V(C_1)$  and to a vertex from  $V(C_2)$ .

**Claim H.1**  $C_1 = T_k$ .

**Proof.** Assume, to the contrary, that  $C_1 = E_k$ , and so  $C_1$  has  $k$  vertices. By our choice of  $C_1$ ,  $\sum_{e \in E_v} |V(C_1) \cap V(e)| \geq 3(k-1)/2$ . Since  $k \geq 5$ , we have that  $3(k-1)/2 > k$ . Hence by the pigeonhole principle, at least one vertex,  $u_1$  say, of  $C_1$  is contained in two edges of

$E_v$ , and so  $u_1$  has degree 3 in  $H$ . Renaming the edges in  $E_v$  if necessary, we may assume that  $u_1 \in V(e_1) \cap V(e_2)$ .

If the edge  $e_3$  intersects  $V(C_2)$ , then let  $u_3 \in V(C_2) \cap V(e_3)$ . By Observation 3(d), there is a  $\tau(H')$ -set  $T$  that contains the set  $\{u_1, u_3\}$ . Since  $\{u_1, u_3\}$  intersects all three edges that contain  $v$  in  $H$ , the set  $T$  is a transversal of  $H$ , and so  $\tau(H) \leq |T| = \tau(H')$ , contradicting Claim E(a). Hence the edge  $e_3$  does not intersect  $V(C_2)$ . Thus,  $V(e_3) \setminus \{v\} \subset V(C_1)$ .

Suppose that both edges  $e_1$  and  $e_2$  intersect  $V(C_2)$ . If all vertices of  $V(C_1) \cap V(e_3)$  have degree 2 in  $H$ , then  $V(e_3) \setminus \{v\} = V(C_1) \setminus \{u_1\}$  and  $V(C_1) \cap V(e_1) = \{u_1\} = V(C_1) \cap V(e_2)$ . Thus,  $3(k-1)/2 \leq \sum_{e \in E_v} |V(C_1) \cap V(e)| = k+1$ , and so  $k \leq 5$ . Consequently,  $k = 5$  and we have equality throughout the Inequality Chain (1). But then all three edges in  $E_v$  intersect  $V(C_1)$  but only two edges in  $E_v$  intersect  $V(C_2)$ , contradicting our choice of  $C_1$  and  $C_2$ . Therefore there is a vertex  $x_1 \in V(C_1) \cap V(e_3)$  that has degree 3 in  $H$ . Renaming the edges  $e_1$  and  $e_2$ , if necessary, we may assume that  $x_1 \in V(e_1)$ . By assumption, the edge  $e_2$  intersect  $V(C_2)$ . Let  $x_2 \in V(C_2) \cap V(e_2)$ . By Observation 3(d), there is a  $\tau(H')$ -set  $T$  that contains the set  $\{x_1, x_2\}$ . Since  $\{x_1, x_2\}$  intersects all three edges that contain  $v$  in  $H$ , the set  $T$  is a transversal of  $H$ , and so  $\tau(H) \leq |T| = \tau(H')$ , contradicting Claim E(a). Hence, at most one of  $e_1$  and  $e_2$  intersects  $V(C_2)$ .

Hence renaming  $e_1$  and  $e_2$ , if necessary, we may assume that  $V(e_1) \setminus \{v\} \subset V(C_1)$ . By the pigeonhole principle, there is a vertex  $w_1 \in V(C_1) \cap V(e_1) \cap V(e_3)$ . Since  $H$  is connected, the edge  $e_2$  intersects  $V(C_2)$ . Let  $w_2 \in V(C_2) \cap V(e_2)$ . By Observation 3(d), there is a  $\tau(H')$ -set  $T$  that contains the set  $\{w_1, w_2\}$ . Since  $\{w_1, w_2\}$  intersects all three edges that contain  $v$  in  $H$ , the set  $T$  is a transversal of  $H$ , and so  $\tau(H) \leq |T| = \tau(H')$ , contradicting Claim E(a). Therefore,  $C_1$  is a generalized triangle  $T_k$ .  $\square$

By Claim H.1, the component  $C_1$  is a generalized triangle  $T_k$ . Renaming the edges  $e_1, e_2, e_3$  if necessary, we may assume that

$$|V(C_1) \cap V(e_1)| \geq |V(C_1) \cap V(e_2)| \geq |V(C_1) \cap V(e_3)|,$$

which implies that

$$|V(C_1) \cap V(e_3)| \leq \frac{1}{3} \sum_{e \in E_v} |V(C_1) \cap V(e)|.$$

Therefore,

$$|V(C_1) \cap V(e_1)| + |V(C_1) \cap V(e_2)| \geq \frac{2}{3} \sum_{e \in E_v} |V(C_1) \cap V(e)| \geq k-1 > \frac{1}{2}(k+1).$$

If  $e_3$  does not intersect  $V(C_2)$ , then neither do the edges  $e_1$  and  $e_2$ , implying that  $H$  is disconnected, a contradiction. Hence,  $e_3$  intersect  $V(C_2)$ , and so  $|V(C_1) \cap V(e_3)| \leq k-2$ . If  $|V(C_1) \cap V(e_2)| \geq 2$ , then  $|V(C_1) \cap V(e_1)| \geq 2$ . But then we contradict Claim F. Therefore,  $|V(C_1) \cap V(e_2)| \leq 1$ , and so  $|V(C_1) \cap V(e_3)| \leq 1$ . Now by our choice of  $C_1$ ,

$$k+1 \leq \frac{3}{2}(k-1) \leq \sum_{e \in E_v} |V(C_1) \cap V(e)| \leq (k-1) + 1 + 1 = k+1.$$

Consequently, we must have equality throughout the above inequality chain. In particular,  $\sum_{e \in E_v} |V(C_1) \cap V(e)| = 3(k-1)/2$ ,  $|V(C_1) \cap V(e_1)| = k-1$  and  $|V(C_1) \cap V(e_2)| = |V(C_1) \cap V(e_3)| = 1$ . But then we have equality throughout the Inequality Chain (1) and all three edges in  $E_v$  intersect  $V(C_1)$  but only two edges in  $E_v$  intersect  $V(C_2)$ , contradicting our choice of  $C_1$  and  $C_2$ . Therefore,  $H'$  has at least three components. This completes the proof of Claim H.  $\square$

We now return to the proof of Theorem 1. By Claim H, the hypergraph  $H'$  has at least three components. Let  $F$  be a bipartite graph with partite sets  $V_1$  and  $V_2$ , where  $V_1 = E_v = \{e_1, e_2, e_3\}$  and where the vertices in  $V_2$  correspond to the components of  $H'$ . Further the edge set of  $F$  is defined as follows: If an edge  $e \in E_v$  intersects a component  $C$  of  $H'$  in  $H$ , then the vertex  $e \in V_1$  is adjacent to the vertex  $C \in V_2$  in  $F$ .

Since  $H'$  has at least three components,  $|V_2| \geq 3$ . Since  $H$  is connected, every component in  $H'$  has a nonempty intersection with at least one edge in  $E_v$ , and so every vertex in  $V_2$  has degree at least 1 in  $F$  and  $N_F(V_1) = V_2$ . Thus if  $S = V_1$ , then  $|N_F(S)| = |V_2| \geq 3 = |S|$ . Since every edge  $e \in E_v$  intersects at least one component of  $H'$  in  $H$ , every vertex in  $V_1$  has degree at least 1 in  $F$ . Thus if  $S \subset V_1$  and  $|S| = 1$ , then  $|N_F(S)| \geq |S|$ . Hence by Hall's Matching Theorem, either  $V_1$  can be matched to a subset of  $V_2$  in  $F$  or  $|N_F(S)| < |S|$  for some subset  $S \subset V_1$  with  $|S| = 2$ .

Suppose that  $V_1$  can be matched to a subset of  $V_2$  in  $F$ . Let  $M_F$  be such a matching in  $F$ . We now name the components in  $H'$  so that  $M_F = \{e_1 C_1, e_2 C_2, e_3 C_3\}$ . Hence for  $i \in \{1, 2, 3\}$ , the edge  $e_i$  intersects the component  $C_i$  of  $H'$  in  $H$ . For  $i \in \{1, 2, 3\}$ , let  $u_i \in V(C_i) \cap V(e_i)$ . By Observation 3(d), there is a  $\tau(H')$ -set  $T$  that contains the set  $\{u_1, u_2, u_3\}$ . Since  $\{u_1, u_2, u_3\}$  intersects all three edges that contain  $v$  in  $H$ , the set  $T$  is a transversal of  $H$ , and so  $\tau(H) \leq |T| = \tau(H')$ , contradicting Claim E(a). Therefore,  $|N_F(S)| < |S|$  for some subset  $S \subseteq V_1$  with  $|S| = 2$ .

Renaming the edges in  $E_v$  if necessary, we may assume that  $S = \{e_1, e_2\}$ . Thus in  $H$  we have that  $V(e_1), V(e_2) \subseteq V(C) \cup \{v\}$  for some component  $C$  of  $H'$ . Since  $H$  is connected, the edge  $e_3$  intersects every component of  $H'$  different from  $C$  in  $H$ . Thus,  $|V(C) \cap V(e_1)| = k-1$ ,  $|V(C) \cap V(e_2)| = k-1$  and  $|V(C) \cap V(e_3)| \leq k-3$ . If  $C$  is a generalized triangle  $T_k$ , then we contradict Claim F. Hence,  $C = E_k$ .

Let  $C'$  be an arbitrary component of  $H'$  different from  $C$ , and let  $u_3 \in V(C') \cap V(e_3)$ . Since  $\sum_{i=1}^2 |V(C) \cap V(e_i)| = 2(k-1) > k$ , by the pigeonhole principle at least one vertex,  $u_1$  say, of  $C$  is contained in both edges  $e_1$  and  $e_2$ . By Observation 3(d), there is a  $\tau(H')$ -set  $T$  that contains the set  $\{u_1, u_3\}$ . Since  $\{u_1, u_3\}$  intersects all three edges that contain  $v$  in  $H$ , the set  $T$  is a transversal of  $H$ , and so  $\tau(H) \leq |T| = \tau(H')$ , contradicting Claim E(a). This completes the proof of Theorem 1.  $\square$

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## References

- [1] V. Chvátal and C. McDiarmid, Small transversals in hypergraphs. *Combinatorica* **12** (1992), 19–26.
- [2] M. A. Henning and C. Löwenstein, Hypergraphs with large transversal number and with edge sizes at least four. *Central European J. Math.* **10**(3) (2012), 1133–1140.
- [3] M. A. Henning and A. Yeo, Hypergraphs with large transversal number and with edge sizes at least three. *J. Graph Theory* **59** (2008), 326–348.
- [4] M. A. Henning and A. Yeo, Strong transversals in hypergraphs and double total domination in graphs. *SIAM Journal of Discrete Mathematics.* **24**(4) (2010), 1336–1355.
- [5] M. A. Henning and A. Yeo, Hypergraphs with large transversal number. *Discrete Math.* **313** (2013), 959–966.
- [6] P. Hall, On representation of subsets. *J. London Math. Soc.* **10** (1935), 26–30.
- [7] D. König, Graphen und Matrizen. *Math. Riz. Lapok* **38** (1931), 116–119.
- [8] F. C. Lai and G. J. Chang, An upper bound for the transversal numbers of 4-uniform hypergraphs. *J. Combin. Theory Ser. B* **50** (1990), 129–133.
- [9] C. E. Shannon, A theorem on colouring the lines of a network. *J. Math. Phys.* **28** (1949), 148–151.
- [10] S. Thomassé and A. Yeo, Total domination of graphs and small transversals of hypergraphs. *Combinatorica* **27** (2007), 473–487.
- [11] Zs. Tuza, Covering all cliques of a graph. *Discrete Math.* **86** (1990), 117–126.
- [12] V. G. Vizing, The chromatic class of a multigraph. *Kibernetika* (Kiev) **1** (1965), 29–39 [in Russian]. English translation: *Cybernetics* **1** (1965), 32–41. 29, 102, 103